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ORIGINAL RESEARCH

Nonlocal fractional functional differential equations with measure of noncompactness in Banach space

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Abstract In this paper, we are concerned with the following fractional functional differential equations with nonlocal initial conditions in Banach space

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f(t, x(t), x_t), \quad t \in [0, T], \\ x(0) &= \phi + g(x). \end{aligned}$$

By virtue of the theory of measure of noncompactness associated with Darbo's fixed point theorem, upon making some suitable assumptions, some existence results of mild solutions are obtained. Moreover the results obtained are utilized to study the existence of solutions to fractional parabolic equations as an illustrative example to show the practical usefulness of the analytical results.

Keywords Fractional functional differential equation · Nonlocal initial condition · Hausdorff measure of noncompactness · Mild solution · Darbo's fixed point theorem

Introduction

In this paper, we are concerned with the nonlocal initial value problem

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f(t, x(t), x_t), \quad t \in [0, T], \\ x(0) &= \phi + g(x), \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in a separable Banach space X,

$$f : [0, T] \times X \times C \to X, \quad g : L^p([0, T], X) \to X,$$

are given X-valued functions. The fractional derivative is understood in the Riemann–Liouville sense. The aim of this paper is to study the existence of mild solutions for the fractional functional differential Eq. (1.1) in a separable Banach space. The technique used here is the measure of noncompactness associated with Darbo's fixed point theorem.

The fractional derivative is understood in the Riemann–Liouville sense. The origin of fractional calculus goes back to Newton and Leibnitz in the seventieth century. One observes that fractional order can be very complex in viewpoint of mathematics and they have recently proved to be valuable in various fields of science and engineering. In fact, one can find numerous applications in electrochemistry, electromagnetism, viscoelasticity, biology and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [1, 2] or to model activator–inhibitor dynamics with anomalous diffusion [3]. For details, see [4–7] and the references therein.

Differential equations of fractional order have appeared in many branches of physics and technical sciences [8, 9]. It has seen considerable development in the last decade, see [3–29] and the references therein. Recently, the existence and uniqueness problem for various fractional differential equations were considered by Ahmad [10], Bhaskarc[11], Lakshmikantham and Leela[12] et al. The nonlocal Cauchy problem was considered by Anguraj, Karthikeyan and N'Guérékata [13], and the importance of nonlocal initial

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conditions in different fields has been discussed in [6, 7] and the references therein.

The nonlocal problem (1.1) was motivated by physical problems. Indeed, the nonlocal initial condition $x(0) = \phi + g(x)$ can be applied in physics with better effect than the classical initial condition $x(0) = \phi$. For this reason, the problem (1.1) has gotten considerable attention in recent years, see [30–32] and the references therein. See also [33–35] and the references therein for recent generalizations of problem (1.1) to various kinds of differential equations.

To the best of our knowledge, the existence of mild solutions for the fractional functional differential Eq. (1.1) with nonlocal initial conditions using the theory of measure of noncompactness is a subject that has not been treated in the literature. Our purpose in this paper is to establish some results concerning the existence of mild solutions for equations that can be modeled in the form (1.1) by virtue of the theory of measure of noncompactness associated with Darbo's fixed point theorem. Upon making some appropriate assumptions, some sufficient conditions for the existence of mild solutions for the fractional functional differential Eq. (1.1) are given. It is worthwhile mentioning that the cases of $T(t)$ or f compact and of f Lipschitz are special cases of our conditions. Also we hope that the concept of measure of noncompactness considered here may be a stimulant for further investigations concerning solutions of fractional differential equations of other types.

The rest of this paper is organized as follows. In "Notations, definitions and auxiliary facts" section, we give some notations, definitions and auxiliary facts. "Main results" section contains the main results of this paper with two existence theorems. An example is given to illustrate our results in "Applications" section.

Notations, definitions and auxiliary facts

Let $(X, \|\cdot\|)$ be a real separable Banach space. Denote by $C([0,T],X)$ the space of X-valued continuous functions on $[0,T]$ and by $L^p([0,T],X)$ the space of X-valued measurable functions on $[0,T]$ with

$$\int_0^T \|x(t)\|^p \mathrm{d}t < \infty,$$

provided with norm

$$\|x\|_p = \left(\int_0^T \|x(t)\|^p \mathrm{d}t\right)^{\frac{1}{p}}.$$

Let r be a given positive real number, if $x : [-r,T] \to X$, define $x_t \in C([-r,0],X)$ by

$$x_t(\theta) = x(t+\theta), \quad \text{for} \quad -r \le \theta \le 0,$$

and denote

$$\|x\|_C = \sup\left\{\|x(t)\| : t \in [-r,0]\right\}, \quad \text{for} \quad x_t \in C([-r,0],X).$$

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1 [36] The fractional integral of order $\alpha > 0$ with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > t_0, \quad \alpha > 0,$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where f is an abstract continuous function and $\Gamma(\alpha)$ is the Gamma function [36].

Definition 2.2 [36] Riemann–Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^t (t-s)^{-\alpha} f(s)\mathrm{d}s, \quad t > t_0, \quad n-1 < \alpha < n.$$

The first and maybe the most important property of Riemann–Liouville fractional derivative is that for $t > t_0$ and $\alpha > 0$, we have

$$D_t^\alpha (I^\alpha f(t)) = f(t),$$

which means that Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the same order α.

Let Y be a Banach space, for bounded set $B \subset Y$, the Hausdorff's measure of noncompactness χ_Y is defined by

$$\chi_Y(B) = \inf\left\{r > 0, B \text{ can be covered by finite number of balls with radii } r\right\}.$$

In this paper, we denote χ by the Hausdorff's measure of noncompactness of X and denote χ_p by the Hausdorff's measure of noncompactness of $L^p([0,T],X)$. To discuss the problem in this paper, we need the following lemmas.

Lemma 2.1 [37] Let $B, C \subset Y$ be bounded, the following properties are satisfied

(1) B is precompact if and only if $\chi_Y(B) = 0$;
(2) $\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(\mathrm{conv}B)$, where $\overline{B}$ and $\mathrm{conv}B$ mean the closure and convex hull of B, respectively;
(3) $\chi_Y(B) \le \chi_Y(C)$ when $B \subseteq C$;
(4) $\chi_Y(B+C) \le \chi_Y(B) + \chi_Y(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;

(5) $\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$;

(6) $\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ for any $\lambda \in \mathbb{R}$;

(7) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then

$$\chi_Z(QB) \leq k\chi_Y(B),$$

for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space;

(8) $\chi_Y(B) = \inf\{d_Y(B, C) : C \subseteq Y$ be precompact$\}$ $= \inf\{d_Y(B, C) : C \subseteq Y$ be finite valued$\}$, where $d_Y(B, C)$ means Hausdorff distance between B and C in Y;

(9) If $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and

$$\lim_{n\to\infty} W_n = 0,$$

then $\cap_{n=1}^{\infty} W_n$ is nonempty and compact in Y. The map $Q : W \subseteq Y \to Y$ is said to be a χ_Y-contraction if there exists a positive constant $k < 1$ such that

$$\chi_Y(Q(C)) \leq k\chi_Y(C),$$

for any bounded closed subset $C \subseteq W$.

In 1955, Darbo [38] proved the fixed point property for α-set contraction (i.e., $\alpha(S(A)) \leq k\alpha(A)$ with $k \in [0, 1]$) on a closed, bounded and convex subset of Banach spaces. Since then many interesting works have appeared. For example, in 1972, Sadovskii [39] proved the fixed point property for condensing functions (i.e., $\alpha(S(A)) < \alpha(A)$ with $\alpha(A) \neq 0$) on closed, bounded and convex subset of Banach spaces. It should be noted that any α-set contraction is a condensing function, but the converse is not true (see [40]). In 2007, Hajji and Hanebaly [41] proved the existence of a common fixed point for commuting mappings satisfying

$$\alpha(S(A)) \leq k \sup_{i\in I}(\alpha(T_i(A))), \quad \alpha(S(A)) < \sup_{i\in I}(\alpha(T_i(A)), \alpha(A)),$$

where α is the measure of noncompactness on a closed, bounded and convex subset Ω of a locally convex space X, T_i and S are continuous functions from Ω to Ω with T_i, and in addition, are affine or linear. Furthermore, for every $i \in I$, T_i are equal to the identity function, moreover the obtain in particular Darbo's (see [38]) as well as Sadovskii's (see [39]) fixed point theorems, which are used to study the existence of solutions of one equation. Recently, Hajji [42] present common fixed point theorems for commuting operators which generalize Darbo's and Sadovskii's fixed point theorems, furthermore, as examples and applications, they study the existence of common solutions of equations in Banach spaces using measure of noncompactness. Our purpose in this paper is to establish some results concerning the existence of mild solutions for equations that can be modeled in the form (1.1) by virtue of the theory of measure of noncompactness associated with Darbo's fixed point theorem.

Lemma 2.2 ([37], Darbo–Sadovskii) If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is a χ_Y-contraction, then Q has at least one fixed point in W.

We call $B \subset L([0, T], X)$ uniformly integrable if there exists $\eta \in L([0, T], \mathbb{R}^+)$ such that

$$\|u(s)\| \leq \eta(s), \quad \text{for all} \quad u \in B \quad \text{and a.e.} \quad s \in [0, T].$$

Lemma 2.3 [43] If $\{u_n\}_{n=1}^{\infty} \in L([0, T], X)$ is uniformly integrable, then $t \to \chi\left(\{u_n(t)\}_{n=1}^{\infty}\right)$ is measurable and

$$\chi\left(\left\{\int_0^t u_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right) \leq \int_0^t \chi\left(\{u_n(s)\}_{n=1}^{\infty}\right)\mathrm{d}s.$$

Lemma 2.4 [44] Let $B \subset C([0, T], X)$ be bounded and equicontinuous on $[0, T]$. Then

$$\chi_p = \left(\int_0^T \chi^p(B(t))\mathrm{d}t\right)^{\frac{1}{p}},$$

where

$$B(t) = \{u(t) : u \in B\} \subset X.$$

A C_0 semigroup $T(t)$ is said to be equicontinuous if

$$t \to \{T(t)x : x \in B\}$$

is equicontinuous for all bounded set B in X and t > 0. It is known that the analytic semigroup is equicontinuous.

The following lemma is obvious.

Lemma 2.5 If the semigroup $T(t)$ is equicontinuous, $\eta \in L([0, T], \mathbb{R}^+)$, then the set

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)u(s)\mathrm{d}s : \|u(s)\| \leq \eta(s)\right\},$$

for a.e. $s \in [0, T]$, is equicontinuous for all $t \in [0, T]$.

Proof Note that,

$$\begin{aligned}
&\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t+h}(t+h-s)^{\alpha-1}T(t+h-s)u(s)\mathrm{d}s\right.\\
&\quad\left.-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{\alpha-1}T(t-s)u(s)\mathrm{d}s\right\|\\
&\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t \left[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]\right.\\
&\quad T(t+h-s)u(s)\mathrm{d}s\|\\
&\quad+\frac{1}{\Gamma(\alpha)}\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}T(t+h-s)u(s)\right\|\mathrm{d}s
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[T(t+h-s)-T(t-s)]u(s)\right\|\mathrm{d}s$$
$$=I+II+III, \tag{2.1}$$

where

$$I=\frac{1}{\Gamma(\alpha)}\left\|\int_0^t\left[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]\right.$$
$$\left.T(t+h-s)u(s)\mathrm{d}s\right\|$$
$$II=\frac{1}{\Gamma(\alpha)}\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}T(t+h-s)u(s)\right\|\mathrm{d}s$$
$$III=\frac{1}{\Gamma(\alpha)}\left\|\int_0^t(t-s)^{\alpha-1}[T(t+h-s)-T(t-s)]u(s)\right\|\mathrm{d}s.$$

Estimating the terms on the right-hand side of (2.1) yields

$$I\le\frac{M}{\Gamma(\alpha)}\int_0^t|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|\eta(s)\mathrm{d}s$$
$$\le\frac{M}{\Gamma(\alpha)}\int_0^{t-\varepsilon}[(t-s)^{\alpha-1}-(t+h-s)^{\alpha-1}]\eta(s)\mathrm{d}s$$
$$+\frac{M}{\Gamma(q)}\int_{t-\varepsilon}^t(t-s)^{\alpha-1}\eta(s)\mathrm{d}s$$
$$=I'+II',$$

where

$$I'=\frac{M}{\Gamma(\alpha)}\int_0^{t-\varepsilon}[(t-s)^{\alpha-1}-(t+h-s)^{\alpha-1}]\eta(s)\mathrm{d}s$$
$$II'=\frac{M}{\Gamma(q)}\int_{t-\varepsilon}^t(t-s)^{\alpha-1}\eta(s)\mathrm{d}s,$$

with

$$M=\sup\{\|T(t)\|:t\in[0,T]\}.$$

It follows from the assumption of $\eta(s)$ that $I'\to0$ as $h\to0$. Using Hölder inequality, one obtains $II'\to0$ as $h\to0$ and $\varepsilon\to0$.

For II, one has

$$II\le\frac{1}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}\|T(t+h-s)u(s)\|\mathrm{d}s$$
$$\le\frac{M}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}\eta(s)\mathrm{d}s\to0 \text{ as } h\to0.$$

As to III, one gets

$$III\le\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t-\varepsilon}(t-s)^{\alpha-1}[T(t+h-s)-T(t-s)]u(s)\right\|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\left\|\int_{t-\varepsilon}^t(t-s)^{\alpha-1}[T(t+h-s)-T(t-s)]u(s)\right\|\mathrm{d}s$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\left\|T\left(\frac{h}{2}+\frac{t+h-s}{2}\right)-T\left(\frac{t-s}{2}\right)\right\|$$
$$\left\|T\left(\frac{t-s}{2}\right)u(s)\right\|\mathrm{d}s$$
$$+\frac{2M}{\Gamma(\alpha)}\int_{t-\varepsilon}^t(t-s)^{\alpha-1}\eta(s)\mathrm{d}s$$
$$\le\frac{M}{\Gamma(\alpha)}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\left\|T\left(\frac{h}{2}+\frac{t+h-s}{2}\right)\right.$$
$$\left.-T\left(\frac{t-s}{2}\right)\right\|\eta(s)\mathrm{d}s+\frac{2M}{\Gamma(\alpha)}\int_{t-\varepsilon}^t(t-s)^{\alpha-1}\eta(s)\mathrm{d}s$$
$$=I''+II'',$$

where

$$I''=\frac{M}{\Gamma(\alpha)}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\left\|T\left(\frac{h}{2}+\frac{t+h-s}{2}\right)\right.$$
$$\left.-T\left(\frac{t-s}{2}\right)\right\|\eta(s)\mathrm{d}s$$
$$II''=\frac{2M}{\Gamma(\alpha)}\int_{t-\varepsilon}^t(t-s)^{\alpha-1}\eta(s)\mathrm{d}s.$$

Using the assumption that $T(t)$ is equicontinuous in X, integrating with $s\to\eta(s)\in L([0,T],\mathbb{R}^+)$, one sees that $I''\to0$ as $h\to0$. From the assumption of $\eta(s)$ and Hölder inequality, it is easy to see that $II''\to0$ as $h\to0$ and $\varepsilon\to0$. Therefore, the family of functions

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^{\cdot}(\cdot-s)^{\alpha-1}T(\cdot-s)u(s)\mathrm{d}s:\|u(s)\|\le\eta(s)\right\},$$

is equicontinuous. □

Main results

In this section, we use the measure of noncompactness of $L^p([0,T],X)$ to consider the following functional differential equations of fractional order $0<\alpha>1$ when g is continuous in the norm of $L^p([0,T],X)$

$$\mathrm{D}^\alpha x(t)=Ax(t)+f(t,x(t),x_t),\quad t\in[0,T],\quad x(0)=\phi+g(x). \tag{3.1}$$

Eq. (3.1) will be considered under the following assumptions:

(H$_1$) The C_0-semigroup $\{T(t)\}_{t\ge0}$ generated by A is equicontinuous;

(H$_2$)

(1) $f:[0,T]\times X\times C([-r,0],X)\to X$ satisfies the Carathéodory-type condition, i.e., $f(\cdot,x,\varphi):[0,T]\to X$

is measurable for all $(t,x,\varphi) \in [0,T] \times X \times C([-r,0],X)$ and $f(t,\cdot) : X \times C([-r,0],X) \to X$ is continuous for a.e. $t \in [0,T]$;

(2) there exists $d_2, e_2 \in L^p([0,T],\mathbb{R}^+)$ such that for all $(t,x,\varphi) \in [0,T] \times X \times C([-r,0],X)$

$$\|f(t,x,\varphi)\| \le d_2(t)(\|x\| + \|\varphi\|_C) + e_2(t);$$

(3) there exists $c_2 \in L^q([0,T],\mathbb{R}^+)$ such that for a.e. $t,s \in [0,T]$ and any bounded subset $D_1 \subseteq X$, $D_2 \subseteq C([-r,0],X)$

$$\chi\Big(T(t)f(s,D_1,D_2)\Big) \le c_2(t) \Big(\chi(D_1) + \sup_{\theta\in[-r,0]} \chi(D_2(\theta))\Big),$$

where

$$D_2(\theta) = \{v(\theta) : v \in D_2\} \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1;$$

(H$_3$)

(1) The function $g : L^p([0,T],X) \to X$ is continuous;

(2) there exist positive constants d_1, e_1 such that

$$\|g(x)\| \le d_1\|x\|_p + e_1, \text{ for any } x \in L^p([0,T],X);$$

(3) there exists a positive constant c_1 such that for any $B \subset C([0,T],X)$ which is bounded and equicontinuous on [0,*T*],

$$\chi\Big(T(t)g(B)\Big) \le c_1\chi_p(B), \text{ for any a.e. } t \in [0,T];$$

(H$_4$)

$$Md_1T^{\frac{1}{p}} + \frac{4MT^{\alpha}\|d_2\|_p}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} < 1.$$

Definition 3.1 A continuous function $x : [-r,T] \to X$ satisfying the integral equation

$$x(t) = T(t)(x(0) - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} T(t-\eta)f(\eta, x(\eta), x_\eta)\mathrm{d}\eta,$$

is called a mild solution for Eq.(3.1).

Now, we are prepared to state and prove our main theorems of this section.

Theorem 3.1 Let (H_1)–(H_4))be satisfied. Then Eq.(3.1) has at least one mild solution whenever

$$T^{\frac{1}{p}}\left(c_1 + \frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right) < 1.$$

Proof For each $k \in \mathbb{N}$, denote by

$$B_k = B_k(L^p([-r,T],X)) = \Big\{x \in L^p([-r,T],X) : \|x(s)\| \le k, s \in [-r,T]\Big\}.$$

Obviously $B_k \subset L^p([-r,T],X)$ is uniformly integrable, closed and convex. For each $x \in B_k$, the restriction of x on $[0,T]$ denoted by $x|_{[0,T]}$ is an element of $B_k(L^p([0,T],X))$. For simplicity, we also write $g(x|_{[0,T]})$ as $g(x)$.

Define $F : L^p([-r,T],X) \to L^p([-r,T],X)$ by $F = F_1 + F_2$, where

$$(F_1x)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ T(t)(x(0) - g(x)), & t \in [0,T], \end{cases}$$

$$(F_2x)(t) = \begin{cases} 0, & t\in[-r,0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\eta)^{\alpha-1}T(t-\eta)f(\eta,x(\eta),x_\eta)\mathrm{d}\eta, & t\in[0,T]. \end{cases}$$

First, we show that F is well defined.

If $t \in [-r,0]$, then

$$\|Fx(t)\| \le \|\phi\|_C,$$

and if $t \in [0,T]$, one has

$$\begin{aligned}
&\|Fx(t)\| \le \|F_1x(t) + F_2x(t)\| \\
&\le M(\|\phi(0)\| + \|g(x(0))\|) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \|(t-\eta)^{\alpha-1}T(t-\eta)f(\eta,x(\eta),x_\eta)\|\mathrm{d}\eta \\
&\le M\Big[\|\phi(0)\| + d_1\|x\|_p + e_1 \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}\big[d_2(\eta)\big(\|x\| + \|x\|_C\big) + e_2(\eta)\big]\mathrm{d}\eta\Big] \\
&\le M(\|\phi(0)\| + d_1\|x\|_p + e_1) + \frac{M}{\Gamma(\alpha)}\left(\int_0^t (t-\eta)^{\frac{(\alpha-1)p}{p-1}}\mathrm{d}s\right)^{\frac{p-1}{p}} \\
&\quad \left(\int_0^t \big(d_2(\eta)(\|x\| + \|x\|_C\|) + e_2(\eta)\big)^p\mathrm{d}\eta\right)^{\frac{1}{p}} \\
&\le M(\|\phi(0)\| + d_1\|x\|_p + e_1) + \frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}T^{\alpha-\frac{1}{p}} \\
&\quad \left(\int_0^t \big(d_2(\eta)(\|x\| + \|x\|_C\|) + e_2(\eta)\big)^p\mathrm{d}\eta\right)^{\frac{1}{p}} \\
&\le M(\|\phi(0)\| + d_1T^{\frac{1}{p}}k + e_1) + \frac{2M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} \\
&\quad T^{\alpha-\frac{1}{p}}\Big[2\|d_2\|_pT^{\frac{1}{p}}k + \|e_2\|_p\Big].
\end{aligned}$$

Thus, one has

$$\|Fx(t)\| \le \max\left\{\|\phi\|_C,\ M(\|\phi(0)\| + d_1 T^{\frac{1}{p}} k + e_1) + \frac{2MT^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\left[2\|d_2\|_p T^{\frac{1}{p}} k + \|e_2\|_p\right]\right\}.$$

Thus, we conclude that Fx exists.

Second, we show that there is a $k \in \mathbb{N}$ such that $F(B_k) \subseteq B_k$.

Suppose contrary that for each $k \in \mathbb{N}$ there is $x^k \in B_k$ and $t^k \in [-r, T]$ such that

$$\|Fx(t^k)\| > k.$$

If $t^k \in [-r, 0]$, then

$$k \le \|Fx(t^k)\| \le \|\phi(t^k) + g(x(t^k))\| \le \|\phi\|_C + d_1 T^{\frac{1}{p}} k + e_1,$$

and if $t^k \in [0, T]$, one has

$$\begin{aligned} k &\le \|Fx(t^k)\| \le \|F_1x(t^k) + F_2x(t^k)\| \\ &\le M(\|\phi(0)\| + \|g(x(0))\|) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t^k} \left\|(t^k-\eta)^{\alpha-1} Q(t^k-\eta) f(\eta, x(\eta), x_\eta)\right\| \mathrm{d}\eta \\ &\le M\Big[\|\phi(0)\| + d_1\|x\|_p + e_1 \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}\left[d_2(\eta)\left(\|x\| + \|x\|_C\right) + e_2(\eta)\right]\mathrm{d}\eta\Big] \\ &\le M(\|\phi(0)\| + d_1 T^{\frac{1}{p}} k + e_1) + \frac{2M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} \\ &\quad T^{\alpha-\frac{1}{p}}\left[\|d_2\|_p T^{\frac{1}{p}} k + \|e_2\|_p\right]. \end{aligned} \tag{3.2}$$

Divided by k on both sides of (3.2), one has

$$1 \le M d_1 T^{\frac{1}{p}} + \frac{2MT^{\alpha}\|d_2\|_p}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}},$$

which contradicts the hypotheses (H_4). Therefore, there is a $k \in \mathbb{N}$ such that $F(B_k) \subseteq B_k$.

From now on, we will restrict F on such B_k.

Third, we will verify that F is a χ_C-contraction.

To this end, from the hypothesises (H_2) (1) and (3), one can prove that F is continuous by the continuity of g and of the operator f. The hypothesis (H_1) and Lemma 2.5 imply that $FB_k \subset C([0, T], X)$ is bounded and equicontinuous on [0, T], so is conv(FB_k). As X is separable, from Lemma 2.1 and Lemma 2.3–2.5 for any $B \subset \mathrm{conv}(FB_k)$, one has

$$\begin{aligned} \chi(FB(t)) &\le \chi(T(t)(x_0 - g(B))) \\ &\quad + \chi\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} T(t-\eta) f(\eta, B(\eta), B_\eta\right) \\ &\le \chi(T(t)(x_0 - g(B))) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \chi((t-\eta)^{\alpha-1} T(t-\eta) f(\eta, B(\eta), B_\eta)\mathrm{d}\eta \\ &\le \chi(T(t)(x_0 - g(B))) + \frac{1}{\Gamma(\alpha)} \\ &\quad \int_0^t (t-\eta)^{\alpha-1}\chi(T(t-\eta) f(\eta, B(\eta), B_\eta)\mathrm{d}\eta \\ &\le c_1\chi_p(B) + \frac{2}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} c_2(\eta)\chi(B)\mathrm{d}\eta \\ &\le \left(c_1 + \frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(B), \end{aligned}$$

for a.e. $t \in [0, T]$, where

$$B(t) = \{x(t) : x \in B\} \subseteq X,\ \ B_t = \{x_t : x \in B\} \subseteq C([-r, 0], X).$$

By Lemma 2.3, this implies that

$$\chi_p(FB) \le T^{\frac{1}{p}}\left(c_1 + \frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(B). \tag{3.3}$$

Note that, by Lemma 2.4, the inequality (3.3) may not remain valid in the case of $B \subset B_k$ as B_k is not equicontinuous on [0,T]. So one must look for another closed convex and bounded subset of $L^p([0, T], X)$ such that F is a χ_p-contraction on it.

Let

$$U = L^p - \mathrm{conv}(FB_k),$$

where $L^p - \mathrm{conv}$ means closure of convex hull in $L^p([0, T], X)$. Then

$$FU \subset U \quad \text{as} \quad FB_k \subset B_k,$$

and B_k is closed and convex in $L^p([0, T], X)$. For any closed subset $V \subset U$, let

$$B = V \cap \mathrm{conv}(FB_k).$$

Then

$$V = L^p - \mathrm{cl}(B),$$

where $L^p - \mathrm{cl}$ means closure in $L^p([0, T], X)$. Furthermore

$$FV \subset L^p - \mathrm{cl}(FB),$$

as F is continuous on $L^p([0, T], X)$. By (3.3) this implies that

$$\begin{aligned} \chi_p(FV) &\le \chi_p(L^p - \mathrm{cl}(FB)) = \chi_p(FB) \\ &\le T^{\frac{1}{p}}\left(c_1 + \frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(B) \end{aligned}$$

$$\le T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(V).$$

Since

$$T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)<1,$$

so $F: U \to U$ is a continuous χ_p-contraction. By Darbo–Sadovskii's fixed point theorem, there is a fixed point x of F on B_k, which is a mild solution of the Eq. (3.1). This completes the proof due to Lemma 2.2.

Remark 3.1 Clearly the conclusion of Theorem 3.1 remains valid if the hypotheses (H_2) (3) and (H_4) (3) are replaced by the following (H_2) (3') and (H_4) (3'), respectively: (H_2) (3') There exists $c_2 \in L^q([0,T],\mathbb{R}^+)$ such that

$$\chi(f(t,D_1,D_2)) \le \frac{c_2(t)}{M}\left(\chi(D_1)+\sup_{\theta\in[-r,0]}\chi(D_2(\theta))\right),$$

for a.e. [0,*T*] and any bounded subset $D_1 \subseteq X$, $D_2 \subseteq C([-r,0],X)$;

(H_4) (3') There exists a positive constant c_1 such that for any $B \subset C([0,T],X)$ which is bounded and equicontinuous on [0,*T*],

$$\chi(g(B)) \le c_1\chi_p(B)/M,\ \text{ for any a. e. } t\in[0,T].$$

Remark 3.2 The hypothesis (H_2) (3') (H_4) (3') holds if $T(t)$ is compact or f (g) is the sum of compact and Lipschitz functions with constant $c_2(s) = M$ $(c_1 = M)$.

If X is a Hilbert space, and ϕ is a proper, convex and lower semicontinuous function from X into $(-\infty,+\infty)$, then its subdifferential $\partial\Phi$ is m-accretive. Let $A = \partial\Phi$ then A generates an equicontinuous nonlinear contraction semigroup (cf. [45, 46]). From above we can get the following existence result.

Corollary 3.1 If X is a separable Hilbert space, the hypotheses (H_2)–(H_4) are true, and $A = \partial\Phi$ with ϕ is proper, convex and lower semicontinuous from X into $(-\infty,+\infty)$. Then the nonlocal Eq. (3.1) has at least one integral solution provided that

$$T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)<1.$$

Let us now formulate an existence result when g is uniformly bounded.

Theorem 3.2 Assume that (H_1), (H_3) and (H_4) are true with $d_1 = 0$, and f satisfies (H_2) (2), (3). In addition, suppose that there exists $\theta \in L^p([0,T],\mathbb{R}^+)$, an increasing function $\Omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x,\varphi)\| \le \theta(t)\Omega(\|x\|+\|\varphi\|_C), \tag{3.4}$$

for $(t,x,\varphi)\in[0,T]\times X\times C([-r,0],X)$. Then Eq.(3.1) has at least one mild solution whenever

$$T^{\frac{1}{p}}\left(c_1+\frac{3\|c_2\|_q T^{q-1}}{\Gamma(q)}\right)<1,$$

$$M(\|\phi\|+e_1)+\frac{MT^{\alpha-\frac{1}{p}}\|\theta(s)\|_p\Omega(2T^{\frac{1}{p}}k)}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\le k.$$

Proof Define

$$W_0=\left\{x\in L^p([0,T],X):\|x(t)\|\le k \text{ for a.e. } t\in[0,T]\right\}.$$

For any $x \in W_0$,

$$\begin{aligned}
\|Fx(t)\| &\le \|F_1x(t)+F_2x(t)\|\\
&\le M(\|\phi(0)\|+\|g(x(0))\|)+\frac{1}{\Gamma(\alpha)}\\
&\quad \int_0^t \|(t-\eta)^{\alpha-1}T(t-\eta)f(\eta,x(\eta),x_\eta)\|\mathrm{d}\eta\\
&\le M\Bigg[\|\phi(0)\|+e_1+\frac{1}{\Gamma(\alpha)}\\
&\quad \int_0^t (t-\eta)^{\alpha-1}\theta(\eta)\Omega(\|x\|+\|x\|_C)\mathrm{d}\eta\Bigg]\\
&\le M(\|\phi(0)\|+e_1)+\frac{MT^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\\
&\quad \left(\int_0^t \theta^p(s)\Omega^p(\|x\|+\|x\|_C\|)\mathrm{d}s\right)^{\frac{1}{p}}\\
&\le M(\|\phi(0)\|+e_1)\\
&\quad +\frac{MT^{\alpha-\frac{1}{p}}\|\theta(s)\|_p\Omega(2T^{\frac{1}{p}}k)}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.5}$$

for any $t \in [0,T]$. This means that $FW_0 \subset W_0$.

Let

$$W_{n+1} = FW_n \quad \text{for} \quad n = 0,1,2,\ldots.$$

Then

$$FW_n \subset C([0,T],X),$$

is bounded and equicontinuous on [0, T]. Furthermore, $W_{n+1} \subset W_n$, because $W_1 \subset W_0$. Hence,

$$\begin{aligned}
\chi(W_{n+1}(t)) &\le \chi(T(t)(x_0-g(W_n)))+\chi\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}\\
&\quad T(t-\eta)f(\eta,W_n(\eta),(W_n)_\eta)\mathrm{d}\eta\Bigg)\\
&\le \chi(T(t)(x_0-g(W_n)))+\frac{1}{\Gamma(\alpha)}\\
&\quad \int_0^t \chi((t-\eta)^{\alpha-1}T(t-\eta)f(\eta,W_n(\eta),(W_n)_\eta))\mathrm{d}\eta
\end{aligned}$$

$$\le c_1\chi_p(W_n)+\frac{2}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}c_2(\eta)\chi(W_n)\mathrm{d}\eta$$
$$\le\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(W_n),$$

for $n=0,1,2,\ldots$ and a.e. $t\in[0,T]$. By Lemma 2.4, this implies that

$$\chi_p(W_{n+1})\le T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(W_n),$$
$$\text{for } n=0,1,2,\ldots.$$

Define

$$\widehat{W}_n = L^p-\overline{\mathrm{conv}}(W_n),\quad \text{for}\quad n=0,1,2,\ldots.$$

Then $\widehat{W}_{n+1}\subset\widehat{W}_n$ because $W_{n+1}\subset W_n$, and furthermore one has

$$\chi_p(\widehat{W}_{n+1})=\chi_p(W_{n+1})$$
$$\le T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(W_n)$$
$$=T^{\frac{1}{p}}\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(\widehat{W}_n),$$

for $n=0,1,2,\ldots$. Lemma 2.1 shows that

$$\widehat{W}=\cap_{n=1}^{\infty}\widehat{W}_n,$$

is nonempty, convex and compact in $L^p([0,T],X)$ and $F\widehat{W}\subset\widehat{W}$. Let

$$U=\overline{\mathrm{conv}}(F\widehat{W}).$$

Then $U\subset C([0,T],X)$ and $FU\subset U$, since

$$U=\overline{\mathrm{conv}}(F\widehat{W})\subset\overline{\mathrm{conv}}(\widehat{W})\subset L^p-\overline{\mathrm{conv}}(\widehat{W})=\widehat{W}.$$

Now we prove that $U\subset C([0,T],X)$ is compact. First, by the hypothesis (H_1) and Lemma 2.5, $F\widehat{W}$ is equicontinuous on [0, T], as

$$g: L^p([0,T],X)\to X,$$

is continuous and $\widehat{W}\subset L^p([0,T],X)$ is compact. Furthermore,

$$\chi(\widehat{W}(t))\le\chi(T(t)(x_0-g(\widehat{W})))+\chi\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}\right.$$
$$\left.T(t-\eta)f(\eta,\widehat{W}(\eta),\widehat{W}_\eta)\mathrm{d}\eta\right)$$
$$\le\chi(T(t)(x_0-g(\widehat{W})))+\frac{1}{\Gamma(\alpha)}$$
$$\int_0^t\chi((t-\eta)^{\alpha-1}T(t-\eta)f(\eta,\widehat{W}(\eta),\widehat{W}_\eta))\mathrm{d}\eta$$
$$\le c_1\chi_p(\widehat{W})+\frac{2}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}c_2(\eta)\chi(\widehat{W})\mathrm{d}\eta$$
$$\le\left(c_1+\frac{2\|c_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)\chi_p(\widehat{W})=0,$$

for any $t\in[0,T]$. Hence

$$F\widehat{W}\subset C([0,T],X),$$

is precompact, and hence so is

$$U\subset C([0,T],X).$$

The proof is complete by Schauder's fixed point theorem.

Remark 3.3 Without hypothesis (H_2) the map F, defined above, may not be continuous from $L^p([0,T],X)$ to itself, since the operator f may fail to be continuous under the growth condition (3.4) above. So we use the fixed point theorem on $C([0,T],X)$ rather than on $L^p([0,T],X)$, as F is obviously continuous from $C([0,T],X)$ to itself.

Applications

In this section, we give an example to illustrate the above results.

Consider the following nonlinear fractional parabolic systems of the form

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}u(t,x)=\mu_1\triangle u(t,x)+F_1(t,u(t,x),u_t(x),v(t,x),v_t(x)),\\
&\qquad t\in(0,T),\ x\in\Omega,\\
&\frac{\partial^\alpha}{\partial t^\alpha}v(t,x)=\mu_2\triangle v(t,x)+F_2(t,u(t,x),u_t(x),v(t,x),v_t(x)),\\
&\qquad t\in(0,T),\ x\in\Omega,\\
&u(0,x)=\varphi_1+g_1(u(t,x),v(t,x)),\quad x\in\Omega,\\
&v(0,x)=\varphi_2+g_2(u(t,x),v(t,x)),\quad x\in\Omega,
\end{aligned}\tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^p$, $p\ge1$, with smooth boundary Γ, μ_1, μ_2 are positive constants,

$$F_1,F_2:\mathbb{R}\times\mathbb{R}\times C([-q,0],\mathbb{R})\to\mathbb{R},$$

are given mappings. Here

$$F_2(t,u(t,x),u_t(x),v(t,x),v_t(x)) = \int_\Omega h_1(t,x,z,u(z),u_t(z))\mathrm{d}z,$$

$$g_2(u(t,x),v(t,x)) = \int_\Omega \int_0^T h_2(t,x,z,u(z))\mathrm{d}t\mathrm{d}z.$$

Let

$$X = L^2(\Omega) \times L^2(\Omega)$$

be endowed with the inner product $<\cdot,\cdot>$ defined by

$$<(u,v),(\bar{u},\bar{v})> \; = \; <u,\bar{u}>_{L^2(\Omega)} + <v,\bar{v}>_{L^2(\Omega)},$$

for each $(u,v),(\bar{u},\bar{v}) \in X$. Obviously X is a separable real Hilbert space. Define $A : D(A) \subset X \to X$ given by

$$A(u,v) = (\mu_1 \triangle u, \mu_2 \triangle v), \quad \text{for each} \quad (u,v) \in D(A)$$

with the domain

$$D(A) = \left\{ (u,v) \in X : \frac{\partial^\alpha}{\partial t^\alpha} u, \frac{\partial^\alpha}{\partial t^\alpha} v \in X, \triangle u, \triangle v \in X \right\}.$$

Now define

$$F : [0,T] \times X \to X \quad \text{and} \quad g : C([0,T],X) \to X$$

by

$$\begin{aligned} F(t,(u,v),(u_t,v_t)) = \Big(& F_1(t,u(t,x),u_t(x),v(t,x),v_t(x)), \\ & F_2(t,u(t,x),u_t(x),v(t,x),v_t(x)) \Big), \\ g(t,(u,v)) = & \Big(g_1(t,(u,v),(u_t,v_t)), g_2(t,(u,v),(u_t,v_t)) \Big), \end{aligned}$$

for

$$(u,v) \in X, (u_t,v_t) \in C([-q,0],X) \times C([-q,0],X),$$

where F_i and g_i are superposition mappings associated with F_i and g_i defined by

$$\begin{aligned} F_i(t,(u,v)) = \{ & h \in L^2(\Omega), h(x) = F_i(t,(u(x),v(x)), \\ & (u_t(x),v_t(x))), \quad \text{a.e. for } x \in \Omega \}, \\ g_i(t,(u,v)) = \{ & h \in L^2(\Omega), h(x) = g_i((u(x),v(x)), \\ & (u_t(x),v_t(x))), \quad \text{a.e. for } x \in \Omega \}. \end{aligned}$$

Observe that Eq.(4.1) may be rewritten as

$$\begin{aligned} & \frac{\mathrm{d}^\alpha \mathrm{U}}{\mathrm{d}t^\alpha} = Au + F(t,U(t),U_t), \quad t \in (0,T), \quad \text{a. e.} \\ & U(0) = \varphi + g(U), \end{aligned} \tag{4.2}$$

where

$$U(t) = (u(t),v(t)), \quad \varphi = (\varphi_1,\varphi_2),$$

while A, F and g are as above.

Suppose that:

(1) There exists $k_2(t) \in L(0,T)$ such that $F_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathédory-type function and, for $u,u',v,v' \in \mathbb{R}$,

$$|F_1(t,u,v) - F_1(t,u',v')| \le k_2(t)(|u-u'|_2 + |v-v'|^2).$$

(2) There exists a constant k_1 such that $g_1 : \mathbb{R} \to \mathbb{R}$ is a Carathédory-type function and, for $u,u',v,v' \in \mathbb{R}$,

$$|g_1(u,v) - g_1(u',v')| \le k_1(|u-u'|_2 + |v-v'|^2).$$

(3) For $i = 1,2$, $h_i : [0,T] \times \Omega \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. In addition:

(i)

$$|h_i(t,x,z,r,s) - h_i(t,x',z,r,s)| \le \omega_i^k(t,x,x',z),$$

for $(t,x,z),(t,x',z) \in [0,T] \times \Omega \times \Omega$ and $|r|,|s| \le k$, where $\omega_i^k \in L([0,T] \times \Omega^3)$ are such that

$$\lim_{x' \to x} \int_\Omega \int_0^T \omega_i^k(t,x,x',z)\mathrm{d}t\mathrm{d}z = 0,$$

uniformly for $x \in \Omega$, $i = 1,2$, and for $t \in [0,T]$

$$\lim_{x' \to x} \int_\Omega \omega_i^k(t,x,x',z)\mathrm{d}z = 0,$$

uniformly for $x \in \Omega$;

(ii)

$$|h_i(t,x,z,r,s| \le \rho_i(t)(|r|^2 + |s|^2)^{\frac{1}{2}} + \omega_i(t,x,z),$$

where $\rho_i \in L(0,T)$ and

$$\delta_i = \int_0^T \int_{\Omega \times \Omega} (\omega_i(t,x,z))^2 \mathrm{d}x\mathrm{d}z\mathrm{d}t < +\infty, \quad i = 1,2.$$

Adapting the arguments given in [47] it is not difficult to show that g satisfies the hypothesis (H_3) with

$$c_1 = k_1, \quad d_1^2 = k_1^2 + 2m(\Omega)\|\rho_1\|_1^2, \quad e_1^2 = 2m(\Omega)\delta_1,$$

and f satisfies the hypothesis (H_2) with

$$c_2(t) = k_2(t), \quad d^2(t) = k_2^2(t) + 2m(\Omega)\rho_2^2(t), \quad \|e_2\|_1^2 = 2m(\Omega)\delta_2,$$

where $m(\Omega)$ means the Lebesgue measure of Ω in $\mathbb{R}^p$. Using Corollary 3.1, we conclude that Eq.(4.1) has at least one generalized solution

$$(u_1, u_2) \in C([0,T], L^2(\Omega) \times L^2(\Omega)),$$

provided that

$$M\left(k_1^2+2m(\Omega)\|\rho_1\|_1^2\right)^{\frac{1}{2}}T^{\frac{1}{p}}+\frac{4MT^{\alpha}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}$$
$$\left(\int_0^T\left(k_2^2(t)+2m(\Omega)\rho_2^2(t)\right)^{\frac{p}{2}}\mathrm{d}t\right)^{\frac{1}{p}}<1,$$

and

$$T^{\frac{1}{p}}\left(k_1+\frac{2\|k_2\|_q T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\right)<1.$$

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